

Article

On the Supersymmetry of the Klein–Gordon Oscillator [†]

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Abstract: The three-dimensional Klein–Gordon oscillator exhibits an algebraic structure known from supersymmetric quantum mechanics. The supersymmetry is unbroken with a vanishing Witten index, and it is utilized to derive the spectral properties of the Klein–Gordon oscillator, which is closely related to that of the nonrelativistic harmonic oscillator in three dimensions. Supersymmetry also enables us to derive a closed-form expression for the energy-dependent Green’s function.

Keywords: Klein–Gordon oscillator; supersymmetric quantum mechanics; Green’s function

1. Introduction

Starting with Galileo’s pendulum experiment [1] in 1602, and with Hook’s law of elasticity [2] from 1678, harmonic oscillators played significant roles in classical physics. More importantly, the harmonic oscillator was the first system to which early quantum theory was successfully applied by Planck [3] in 1900 when developing his law of black body radiation. Nowadays, the harmonic oscillator is a standard part of any introductory text book on nonrelativistic quantum mechanics. In relativistic quantum mechanics, the harmonic oscillator was initially studied within Dirac’s theory of electrons in the 1960s [4–6], but attracted considerable attention only with the seminal work by Moshinsky and Szczepaniak [7] (see also Quesne and Moshinsky [8]). Inspired by this so-called Dirac oscillator, the Klein–Gordon oscillator (KGO) was studied by various authors [9–11].

The KGO Hamiltonian characterises a relativistic spin-zero particle with mass m minimally coupled to a complex linear vector potential. Since its introduction, the KGO has attracted much interest. The spectral properties of the one-dimensional system were discussed, for example, in [12,13]. For a treatment in noncommutative space, see [14,15]; for recent results in a nontrivial topology, see [16–19] and the references therein.

Since 1990, the Dirac oscillator has been known to exhibit a supersymmetric (SUSY) structure that, in turn, allows for explicit solutions [20–23]. More recently, SUSY also enabled us to formulate Feynman’s path integral approach for Dirac systems [24]. SUSY in the current context is not based on the original idea, which transforms between states with different internal spin-degree of freedom, but refers to what is commonly known nowadays as supersymmetric quantum mechanics; see, for example, [25] and the references therein.

The purpose of the present work is twofold. First, we show that the Klein–Gordon oscillator possesses a hidden SUSY in the aforementioned sense. Second, we derive an explicit expression for the Green function of the KGO. In doing so, we closely follow the generic approach for SUSY in relativistic Hamiltonians with fixed but arbitrary spin [26].

In the next section, we set up the stage with a brief discussion on the KGO Hamiltonian in three space dimensions and show that this Hamiltonian exhibits a SUSY structure by mapping it onto a quantum mechanical SUSY system. This is then utilized to derive explicit results of the system. In Section 3, we derive the eigenvalues and associated eigenstates. In Section 4, we derive the corresponding Green’s function in a closed form. Lastly, Section 5 closes with a summary and some comments.



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2. Supersymmetry

The Hamiltonian form of the Klein–Gordon equation with arbitrary vector potential was originally introduced by Feshbach and Villars [27], from which the KGO Hamiltonian may be constructed via minimal coupling $\vec{p} \rightarrow \vec{\pi} := \vec{p} - im\omega\vec{r}$, where $m > 0$ stands for the mass of the spinless Klein–Gordon particle, and $\omega > 0$ is a coupling constant to be identified with the harmonic oscillator frequency. This minimal coupling might be interpreted as a complex-valued vector potential of form $\vec{A}(\vec{r}) := i(mc\omega/q)\vec{r}$, with q being the particle charge, and c the speed of light. However, such a vector potential is not linked to any kind of gauge invariance, as $\vec{A}(\vec{r}) = i(mc\omega/2q)\vec{\nabla}r^2$ cannot be gauged away by a pure phase factor in the wave function due to the presence of the imaginary unit.

First explicit expressions of the KGO Hamiltonian were presented by Debergh et al. [9] for an isotropic system. The KGO Hamiltonian for a more general anisotropic oscillator system is due to Bruce and Minning [10]. For the sake of simplicity, we consider the isotropic system characterised by Hamiltonian

$$\mathcal{H} := \frac{\vec{\pi}^\dagger \cdot \vec{\pi}}{2m} \otimes (\tau_3 + i\tau_2) + mc^2 \otimes \tau_3 \tag{1}$$

acting on Hilbert space $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$. In the above, τ_i s stand for Pauli matrices

$$\tau_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad i\tau_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

These Pauli matrices do not represent a spin degree of freedom. The 2-spinors on which the above Hamiltonian acts are those originally introduced by Feshbach and Villars [27].

The KGO Hamiltonian (1) is pseudo-Hermitian [28,29], that is, $\mathcal{H}^\dagger = \tau_3 \mathcal{H} \tau_3$, and reads in an explicit 2×2 matrix notation

$$\mathcal{H} = \begin{pmatrix} M & A \\ -A & -M \end{pmatrix}, \tag{3}$$

where we set $M := H_{\text{NR}} + mc^2$ and $A := H_{\text{NR}}$, both in essence being represented by the nonrelativistic harmonic oscillator Hamiltonian in three dimensions

$$H_{\text{NR}} := \frac{\vec{\pi}^\dagger \cdot \vec{\pi}}{2m} = \frac{1}{2m}\vec{p}^2 + \frac{m}{2}\omega^2\vec{r}^2 - \frac{3}{2}\hbar\omega. \tag{4}$$

Here and in the following, we use calligraphic symbols for operators acting on the full Hilbert space $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$, and operators represented in italics act on subspace $L^2(\mathbb{R}^3)$.

Obviously, diagonal and off-diagonal elements in (3) commute, i.e., $[M, A] = 0$. Hence, following the general approach of [26], an $N = 2$ SUSY structure can be established as follows.

$$\begin{aligned} \mathcal{H}_{\text{SUSY}} &:= \frac{1}{2mc^2}(\mathcal{M}^2 - \mathcal{H}^2) = \frac{1}{2mc^2}H_{\text{NR}}^2 \otimes 1, \\ \mathcal{Q} &:= \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad \mathcal{W} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \tau_3, \end{aligned} \tag{5}$$

where we set $\mathcal{M} := M \otimes 1$. The above SUSY operators obey SUSY algebra

$$\begin{aligned} \mathcal{H}_{\text{SUSY}} &= \{\mathcal{Q}, \mathcal{Q}^\dagger\}, \quad \mathcal{Q}^2 = 0 = \mathcal{Q}^{\dagger 2}, \\ [\mathcal{W}, \mathcal{H}_{\text{SUSY}}] &= 0, \quad \{\mathcal{Q}, \mathcal{W}\} = 0 = \{\mathcal{Q}^\dagger, \mathcal{W}\}. \end{aligned} \tag{6}$$

In the current context, the third Pauli matrix plays the role of the Witten parity operator \mathcal{W} . Therefore, the upper and lower components of a general 2-spinor belong to the subspace with positive and negative Witten parity, respectively. We further remark that

$\dim \ker Q = \dim \ker Q^\dagger = \dim \ker H_{NR} = 1$. That is, SUSY is unbroken, as \mathcal{H}_{SUSY} has zero-energy eigenstates [25], but Witten index Δ still vanishes as

$$\Delta := \text{ind } Q = \dim \ker Q - \dim \ker Q^\dagger = 0. \tag{7}$$

To the best of our knowledge, this is the first quantum mechanical system with an unbroken $N = 2$ SUSY but vanishing Witten index, implying that the spectrum of \mathcal{H} is fully symmetric with respect to the origin, as we see in the following section.

3. Spectral Properties

As was recently shown [26], SUSY in a relativistic Hamiltonian implies the existence of a Foldy–Wouthuysen transformation, which brings that Hamiltonian into a block-diagonal form. In the case of the KGO, this transformation operator \mathcal{U} , which is a pseudounitary operator in the sense that $\mathcal{U}^{-1} = \tau_3 \mathcal{U}^\dagger \tau_3$, reads

$$\mathcal{U} := \frac{|\mathcal{H}| + \tau_3 \mathcal{H}}{\sqrt{2(\mathcal{H}^2 + \mathcal{M}|\mathcal{H}|)}} \tag{8}$$

leading to block-diagonal Foldy–Wouthuysen Hamiltonian

$$\mathcal{H}_{FW} := \mathcal{U} \mathcal{H} \mathcal{U}^{-1} = H_{FW} \otimes \tau_3, \quad H_{FW} := \sqrt{2mc^2 H_{NR} + m^2 c^4}. \tag{9}$$

To be more explicit, let us define $\tanh \Theta := A/M = H_{NR}/(H_{NR} + mc^2)$, which then allows for us to write transformation (8) in matrix form [9]

$$\mathcal{U} = \begin{pmatrix} \cosh \frac{\Theta}{2} & \sinh \frac{\Theta}{2} \\ \sinh \frac{\Theta}{2} & \cosh \frac{\Theta}{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{M}{H_{FW}} + 1} & \sqrt{\frac{M}{H_{FW}} - 1} \\ \sqrt{\frac{M}{H_{FW}} - 1} & \sqrt{\frac{M}{H_{FW}} + 1} \end{pmatrix}. \tag{10}$$

The above expressions are functions of operators of which all may be expressed in terms of H_{NR} . Hence, using the spectral theorem, these are well-defined. In fact, with the spectral properties of the nonrelativistic harmonic-oscillator Hamiltonian (4), one can directly obtain those of (1). Let $\psi_{n\ell\mu}$ denote the well-known eigenfunctions of H_{NR} corresponding to eigenvalue $\varepsilon_{n\ell}$; then, we have

$$\begin{aligned} H_{NR} \psi_{n\ell\mu} &= \varepsilon_{n\ell} \psi_{n\ell\mu}, & \varepsilon_{n\ell} &= \hbar\omega(2n + \ell), & n, \ell &\in \mathbb{N}_0, \\ \psi_{n\ell\mu}(\vec{r}) &= \left(\frac{m\omega}{\hbar}\right)^{\ell/2+3/4} \sqrt{\frac{2n!}{\Gamma(n + \ell + 3/2)}} r^\ell e^{-m\omega r^2/\hbar} L_n^{\ell+1/2}\left(\frac{m\omega}{\hbar} r^2\right) Y_{\ell\mu}(\vec{e}), & (11) \\ \mu &\in \{-\ell, -\ell + 1, \dots, \ell - 1, \ell\}, & r &:= |\vec{r}|, & \vec{e} &:= \vec{r}/r, \end{aligned}$$

where $L_n^{\ell+1/2}$ and $Y_{\ell\mu}$ denote the associated Laguerre polynomials and spherical harmonics, respectively. See, for example, ref. [30]. The eigenvalues and eigenfunctions of (1) are explicitly given by

$$\begin{aligned} \mathcal{H} \Psi_{n\ell\mu}^\pm &= E_{n\ell}^\pm \Psi_{n\ell\mu}^\pm, & E_{n\ell}^\pm &= \pm mc^2 \sqrt{1 + \frac{2\varepsilon_{n\ell}}{mc^2}}, & (12) \\ \Psi_{n\ell\mu}^+(\vec{r}) &= \psi_{n\ell\mu}(\vec{r}) \begin{pmatrix} \cosh \frac{\vartheta_{n\ell}}{2} \\ -\sinh \frac{\vartheta_{n\ell}}{2} \end{pmatrix}, & \Psi_{n\ell\mu}^-(\vec{r}) &= \psi_{n\ell\mu}(\vec{r}) \begin{pmatrix} -\sinh \frac{\vartheta_{n\ell}}{2} \\ \cosh \frac{\vartheta_{n\ell}}{2} \end{pmatrix}, \end{aligned}$$

where $\tanh \vartheta_{n\ell} := \varepsilon_{n\ell}/(\varepsilon_{n\ell} + mc^2)$. These states form an orthonormal basis in $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ with respect to the scalar product [27].

$$\langle \Psi_1 | \Psi_2 \rangle := \int_{\mathbb{R}^3} d^3\vec{r} \bar{\Psi}_1(\vec{r}) \tau_3 \Psi_2(\vec{r}), \tag{13}$$

where the overbar stands for the transposed and complex conjugated 2-spinor. That is,

$$\langle \Psi_{n\ell\mu}^\pm | \Psi_{n'\ell'\mu'}^\pm \rangle = \pm \delta_{nn'} \delta_{\ell\ell'} \delta_{\mu\mu'}, \quad \langle \Psi_{n\ell\mu}^\pm | \Psi_{n'\ell'\mu'}^\mp \rangle = 0. \tag{14}$$

Obviously, the scalar product (13), which was already introduced by Feshbach and Villars [27], is not positive definite and might raise some questions on its probabilistic interpretation. However, Mostafazadeh’s theory of pseudo-Hermitian operators provides a solution for this obstacle. The Klein–Gordon case was explicitly discussed in [28,29].

The SUSY ground states associated with nonrelativistic eigenvalue $\varepsilon_{00} = 0$ are given by

$$\Psi_{000}^+(\vec{r}) = \left(\frac{m\omega}{\hbar\pi}\right)^{3/4} e^{-m\omega r^2/\hbar} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Psi_{000}^-(\vec{r}) = \left(\frac{m\omega}{\hbar\pi}\right)^{3/4} e^{-m\omega r^2/\hbar} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{15}$$

with corresponding eigenvalues $E_{00}^\pm = \pm mc^2$. The Foldy–Wouthuysen Hamiltonian (9) can be written as

$$\mathcal{H}_{FW} = mc^2 \sqrt{1 + \frac{2H_{NR}}{mc^2}} \otimes \tau_3, \tag{16}$$

a form already observed for other relativistic Hamiltonians exhibiting a SUSY [26].

4. Green’s Function

The SUSY established for the KGO in the previous section also allows for us to study Green’s function associated with the KGO Hamiltonian (1). Following the general approach of [26], Green’s function, defined by

$$\mathcal{G}(z) := \frac{1}{\mathcal{H} - z}, \quad z \in \mathbb{C} \setminus \text{spec } \mathcal{H}, \tag{17}$$

can be expressed in terms of iterated Green’s function \mathcal{G}_I , that is,

$$\mathcal{G}(z) = (\mathcal{H} + z)\mathcal{G}_I(z^2), \quad \mathcal{G}_I(z^2) := \frac{1}{\mathcal{H}^2 - z^2}. \tag{18}$$

Noting that $\mathcal{H}^2 = 2mc^2(H_{NR} + mc^2/2) \otimes 1$, the iterated Green’s function can be written in terms of nonrelativistic Green’s function $G_{NR}(\varepsilon) := (H_{NR} - \varepsilon)^{-1}$ associated with H_{NR} , as follows.

$$\mathcal{G}_I(z^2) = \frac{1}{2mc^2} G_{NR}(\varepsilon) \otimes 1, \quad \varepsilon := \frac{z^2}{2mc^2} - \frac{mc^2}{2} = 2mc^2 \left(\left(\frac{z}{2mc^2}\right)^2 - \frac{1}{4} \right). \tag{19}$$

Inserting this into above relation (18) results in

$$\mathcal{G}(z) = \frac{1}{2mc^2} \begin{pmatrix} (H_{NR} + mc^2 + z) G_{NR}(\varepsilon) & H_{NR} G_{NR}(\varepsilon) \\ -H_{NR} G_{NR}(\varepsilon) & -(H_{NR} + mc^2 - z) G_{NR}(\varepsilon) \end{pmatrix}. \tag{20}$$

Using defining relation $H_{NR} G_{NR}(\varepsilon) = \varepsilon G_{NR}(\varepsilon)$ with the second relation in (19) leads us to closed-form expression

$$\mathcal{G}(z) = G_{NR}(\varepsilon) \begin{pmatrix} \left(\frac{1}{2} + \frac{z}{2mc^2}\right) \left(\frac{1}{2} + \frac{z}{2mc^2}\right) & \left(\frac{1}{2} + \frac{z}{2mc^2}\right) \left(\frac{z}{2mc^2} - \frac{1}{2}\right) \\ \left(\frac{1}{2} + \frac{z}{2mc^2}\right) \left(\frac{1}{2} - \frac{z}{2mc^2}\right) & \left(\frac{1}{2} - \frac{z}{2mc^2}\right) \left(\frac{z}{2mc^2} - \frac{1}{2}\right) \end{pmatrix}, \tag{21}$$

with ε as defined in (19). The reader is invited to verify that $\mathcal{H} \mathcal{G}(z) = z \mathcal{G}(z)$. With definition $\tanh \vartheta := \varepsilon / (\varepsilon + mc^2)$, the above result may be placed into form

$$\mathcal{G}(z) = \frac{G_{NR}(\varepsilon)}{\cosh \frac{\vartheta}{2} - \sinh \frac{\vartheta}{2}} \begin{pmatrix} \cosh^2 \frac{\vartheta}{2} & \cosh \frac{\vartheta}{2} \sinh \frac{\vartheta}{2} \\ -\cosh \frac{\vartheta}{2} \sinh \frac{\vartheta}{2} & -\sinh^2 \frac{\vartheta}{2} \end{pmatrix}. \tag{22}$$

The coordinate representation of $G_{\text{NR}}(\vec{r}, \vec{r}', \varepsilon) := \langle \vec{r} | G_{\text{NR}}(\varepsilon) | \vec{r}' \rangle$ has been known for long (see, for example [31]) and explicitly reads

$$G_{\text{NR}}(\vec{r}, \vec{r}', \varepsilon) = \frac{1}{rr'} \sum_{\ell=0}^{\infty} G_{\ell}(r, r', \varepsilon) \sum_{\mu=-\ell}^{\ell} Y_{\ell\mu}^*(\vec{e}') Y_{\ell\mu}(\vec{e}), \quad (23)$$

$$G_{\ell}(r, r', \varepsilon) = -\frac{\Gamma\left(\frac{\ell}{2} - \frac{\varepsilon}{2\hbar\omega}\right)}{\sqrt{rr'} \hbar\omega} W_{\lambda, \nu}\left(r_{>}^2 m\omega/\hbar\right) M_{\lambda, \nu}\left(r_{<}^2 m\omega/\hbar\right),$$

where $W_{\lambda, \nu}$ and $M_{\lambda, \nu}$ denote Whittaker's functions, and we set $\lambda := \frac{\varepsilon}{2\hbar\omega} + \frac{3}{4}$, $\nu := \frac{\ell}{2} + \frac{1}{4}$, $r_{>} := \max\{r, r'\}$ and $r_{<} := \min\{r, r'\}$.

5. Summary and Outlook

In this work, we showed that the KGO exhibits a SUSY structure, closely following the general approach of [26]. The SUSY of the KGO was found to be unbroken but with a vanishing Witten index. Despite eigenvalues in (12) having been known for a long time (see, for example, [9,10]), the associated eigenstates in (12) have, to our knowledge, never been presented. In [10] only the eigenstates of \mathcal{H}_{FW} were given. In addition, SUSY enabled us to calculate the KGO Green function in a closed form.

Obviously, the current discussion for an isotropic oscillator may be extended to that for the anisotropic oscillator following Bruce and Manning [10]. Here, in essence, one needs to reduce the problem to three one-dimensional harmonic oscillators. An explicit expression for Green's function may also be obtained, as the one-dimensional harmonic oscillator Green's function is also known in closed form. See, for example, Glasser and Nieto [32], and the discussion of the associated Dirac problem [33]. One may also pursue the path integral approach for the KGO along the lines of the corresponding approach for the Dirac oscillator [24]. Another route for further investigation would be to look at generalised nonharmonic oscillators characterized by a potential function $U_a(r) := \lambda_a r^a$ using minimal substitution $\vec{\pi} := \vec{p} - i(\vec{\nabla} U_a)(r)$. Such power-law potentials obey duality symmetry in classical and nonrelativistic quantum mechanics (see recent work [34] and references therein). In particular, a harmonic potential where $a = 2$, which corresponds to the discussed KGO case, is dual to the Kepler potential where $a = -1$.

Another extension is to apply the current SUSY construction to the relativistic $S = 1$ oscillator. However, as was argued by Debergh et al. [9], the diagonal and off-diagonal matrix elements of the associated Hamiltonian no longer commute. Hence, it may not be possible to establish a SUSY structure.

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